MECHANICAL ENGINEERING DEPARTMENT
UNITED STATES NAVAL ACADEMY

EM423 - INTRODUCTION TO MECHANICAL VIBRATIONS

CONTINUOUS SYSTEMS - FLEXURAL VIBRATION OF BEAMS
PART 2: CLASSICAL SOLUTION

SYMBOLS

r	Mass density
A_x	Cross sectional area
M	Bending moment
V	Internal shear force
I	Second moment of area
x	Distance along the beam
k	Wave number

INTRODUCTION

The first beam vibration handout derived the wave equation for flexural vibration of beams. The solution was a wave solution, which included 4 different waves. This handout takes the beam wave equation and solves it in a form more amenable to determining natural frequencies and displaced shapes. This handout commences with the equation of motion, derived in the previous handout.

$$EI\frac{\partial^4 y}{\partial x^4} + rA_x\frac{d^2 y}{dt^2} = 0$$

Remember, this equation assumes the beam is uniform in all respects along its length. For a harmonic solution with respect to time

$$y(x,t) = Y(x)e^{iwt}$$

The equation of motion becomes:

$$\frac{\partial^4 Y}{\partial x^4} - k^4 Y = 0 \quad \text{with} \quad k^4 = \frac{rA_x w^2}{EI} \text{ (see the first beam handout)}$$

The general solution we choose to use for this handout is:

$$Y(x) = e^{ax}$$

hence

$$\left(a^4 - k^4\right)Y = 0$$

and this is satisfied when

$$a = \pm k \quad \pm \mathrm{i}k$$

Now Euler gives:

$$e^{\pm kx} = \cosh(kx) \pm \sinh(kx)$$
$$e^{\pm ikx} = \cos(kx) \pm i.\sin(kx)$$

which means we can rewrite the general solution in the form:

$$\boxed{y(x,t) = \{B_1 \cosh(kx) + B_2 \sinh(kx) + B_3 \cos(kx) + B_4 \sin(kx)\} e^{iwt}}$$

where B_1*,* B_2*,* B_3 *and* B_4 are determined by the boundary conditions, and may be complex to include phase effects.

The spatial part of the solution (mode shape) only is:

$$\boxed{Y(x) = B_1 \cosh(kx) + B_2 \sinh(kx) + B_3 \cos(kx) + B_4 \sin(kx)}$$

To solve this equation, we apply the boundary conditions at each end of the beam, i.e. at x = 0 and at x = L. The table of common boundary conditions is in the Beams Handout Part 1.

EXAMPLE

As an example, let's consider a cantilever, which is a beam fixed at x = 0 and free at x = L. We wish to determine its natural frequencies, and the displaced shape at resonance (the mode shape).

$$\text{at } x = 0 \text{ (fixed)} \begin{cases} \text{y=0} \\ \dfrac{\partial \text{y}}{\partial \text{x}} = 0 \end{cases} \qquad \text{at } x = L \text{ (free) } \begin{cases} M = 0 \;\text{ or }\; \dfrac{\partial^2 y}{\partial x^2} = 0 \\ V = 0 \;\text{ or }\; \dfrac{\partial^3 y}{\partial x^3} = 0 \end{cases}$$

These conditions require us to differentiate the general solution:

$$y(x,t) = \{B_1 \cosh(kx) + B_2 \sinh(kx) + B_3 \cos(kx) + B_4 \sin(kx)\} e^{iwt}$$
$$\frac{\partial y}{\partial x} = \{B_1 \sinh(kx) + B_2 \cosh(kx) - B_3 \sin(kx) + B_4 \cos(kx)\} k.e^{iwt}$$
$$\frac{\partial^2 y}{\partial x^2} = \{B_1 \cosh(kx) + B_2 \sinh(kx) - B_3 \cos(kx) - B_4 \sin(kx)\} k^2.e^{iwt}$$
$$\frac{\partial^3 y}{\partial x^3} = \{B_1 \sinh(kx) + B_2 \cosh(kx) + B_3 \sin(kx) - B_4 \cos(kx)\} k^3.e^{iwt}$$

Substituting the boundary conditions into these equations yields these four simultaneous equations:

$$(y)_{x=0} = B_1 + B_3 = 0$$

$$\frac{1}{k}\left(\frac{\partial y}{\partial x}\right)_{x=0} = B_2 + B_4 = 0$$

$$\frac{1}{k^2}\left(\frac{\partial^2 y}{\partial x^2}\right)_{x=L} = B_1 \cosh(kL) + B_2 \sinh(kL) - B_3 \cos(kL) - B_4 \sin(kL) = 0$$

$$\frac{1}{k^3}\left(\frac{\partial^3 y}{\partial x^3}\right)_{x=L} = B_1 \sinh(kL) + B_2 \cosh(kL) + B_3 \sin(kL) - B_4 \cos(kL) = 0$$

Rearranging these equations gives the result:

$$\frac{(\cosh(kL) + \cos(kL))}{(\sinh(kL) - \sin(kL))} = \frac{(\sinh(kL) + \sin(kL))}{(\cosh(kL) + \cos(kL))}$$

which can be reduced to:

$$\cosh(kL).\cos(kL) + 1 = 0$$

Remember that k is a function of frequency, so this equation can be used to determine the natural frequencies. It is called the frequency equation, or the characteristic equation. It is usually solved using numerical methods.

The solutions of the frequency equations for many common combinations of boundary conditions are tabulated in several standard vibration texts. They are usually tabulated in non dimensional form as the values of $(k_n L)$ or $(k_n L)^2$. From the definition of the wave number, k:

$$k^4 = \frac{r A_x w^2}{EI}$$

we get

$$w_n = (k_n L)^2 \sqrt{\frac{EI}{r A_x L^4}}$$

The natural frequencies for a specific beam can now be calculated from the tabulated values of $(k_n L)^2$.

$$\boxed{\begin{aligned} w_n &= (k_n L)^2 \sqrt{\frac{EI}{r A_x L^4}} \quad \text{rad/s} \\ f_n &= \frac{(k_n L)^2}{2p} \sqrt{\frac{EI}{r A_x L^4}} \quad \text{Hz} \end{aligned}}$$

	Fundamental	2nd Mode	3rd Mode	4th Mode	5th Mode
Pin-Pin	9.870	39.48	88.83	157.9	246.7
Cantilever (fix-free)	3.516	22.03	61.70	120.9	199.9
Free-Free	22.37	61.67	120.9	199.9	298.6
Fixed-Fixed	22.37	61.67	120.9	199.9	298.6
Fixed-Pinned	15.418	49.96	104.25	178.27	272.0

Selected values of $(k_n L)^2$

MODE SHAPE

We now consider the mode shape, or deflected shape at resonance. We find this by solving the boundary condition equations for B_1, B_2, B_3 and B_4. These terms are then substituted back into the general solution. For the cantilever considered here, the solution for the mode shape is:

$$Y(x) = \left(\cosh(kx) - \cos(kx)\right) - \left(\frac{\left(\sinh(kL) - \sin(kL)\right)}{\left(\cosh(kL) + \cos(kL)\right)}\right)\left(\sinh(kx) - \sin(kx)\right)$$

Remember that the total actual motion of the beam is a function of both position along the beam, and time.

$$y(x,t) = \left\{\left(\cosh(kx) - \cos(kx)\right) - \left(\frac{\left(\sinh(kL) - \sin(kL)\right)}{\left(\cosh(kL) + \cos(kL)\right)}\right)\left(\sinh(kx) - \sin(kx)\right)\right\}\sin(wt)$$

Again, standard texts have these mode shape equations tabulated for most common pairs of boundary conditions.

ASSIGNMENTS

1. A concrete test beam 2 in × 2 in × 12 in was rigidly supported at two points, 0.224L from each end. The beam resonated at 1690 Hz. If the volumetric weight density of the concrete was ρ = 153 lb/ft3, determine Young's Modulus of Elasticity, E.

2. Determine the characteristic equation for a uniform beam, length = L, clamped at both ends.

3. (extra credit) A uniform beam of length = L and weight = W_B is clamped at one end and carries a concentrated weight W_O at the other end. State the boundary conditions and determine the frequency equation.

SOLUTIONS

1. A concrete test beam 2 in × 2 in × 12 in was rigidly supported at two points, 0.224L from each end. The beam resonated at 1690 Hz. If the volumetric weight density of the concrete was ρ = 153 lb/ft3, determine Young's Modulus of Elasticity, E.

from the handout, for *n* = 1 for a free-free beam, $(k_nL)^2$ = 22.37.

$$\left(k_nL\right)^4 = \frac{rA_X w^2 L^4}{EI} \qquad \text{hence} \qquad E = \frac{rA_X w^2 L^4}{\left(k_nL\right)_4 I}$$

$$rA_X = \frac{153 \times \frac{2}{12} \times \frac{2}{12}}{32.2} \qquad I = \frac{\frac{2}{12} \times \left(\frac{2}{12}\right)^3}{12}$$

$$w^2 = \left(1690 \times 2p\right)^2 \qquad L = \frac{12}{12}$$

$$\text{hence} \qquad E = \frac{153 \times 12^3 \times \left(1690 \times 2p\right)^2}{32.2 \times \left(22.37\right)^2 \times 2^2} = 463 \times 10^6 \text{ lb/ft}^2$$

2. Determine the characteristic equation for a uniform beam, length = L, clamped at both ends.

$$y\left(x,t\right) = \left\{A.\cosh\left(kx\right) + B.\sinh\left(kx\right) + C.\cos\left(kx\right) + D.\sin\left(kx\right)\right\}\sin\left(wt\right)$$

<u>At x = 0</u>

$$\left.\begin{array}{lll} \text{y=0} & \text{so} & \text{A+C=0} \\ \frac{\partial y}{\partial x} & \text{so} & \text{B+D=0} \end{array}\right\} \begin{array}{l} C = -A \\ D = -B \end{array}$$

<u>At x = L</u>

$$\text{y=0} \qquad \text{so} \qquad \text{A}\left\{\cosh\left(\text{kL}\right) - \cos\left(kL\right)\right\} + B\left\{\sinh\left(kL\right) - \sin\left(kL\right)\right\} = 0$$

$$\frac{\partial y}{\partial x} \qquad \text{so} \qquad A\left\{\sinh\left(kL\right) + \sin\left(kL\right)\right\} + B\left\{\cosh\left(kL\right) - \cos\left(kL\right)\right\} = 0$$

$$-\frac{A}{B} = \frac{\sinh\left(kL\right) - \sin\left(kL\right)}{\cosh\left(kL\right) - \cos\left(kL\right)} = \frac{\cosh\left(kL\right) - \cos\left(kL\right)}{\sinh\left(kL\right) + \sin\left(kL\right)}$$

or

$$\cosh\left(kL\right)\cos\left(kL\right) = 1$$

3. (extra credit) A uniform beam of length = L and weight = W_B is clamped at one end and carries a concentrated weight W_O at the other end. State the boundary conditions and determine the frequency equation.

The solution to extra credit problems is available from your instructor.